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**Sensitivity Analysis of the Simultaneous  
Proportionate Change of Inputs and Outputs  
in Data Envelopment Analysis**

by

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# SENSITIVITY ANALYSIS OF THE SIMULTANEOUS PROPORTIONATE CHANGE OF INPUTS AND OUTPUTS IN DATA ENVELOPMENT ANALYSIS\*

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**Abstract:** Sensitivity analysis in data envelopment analysis for the Charnes-Cooper-Rhodes ratio model is studied for the case of the simultaneous proportionate increase of all inputs and proportionate decrease of all outputs of an efficient decision making unit for which efficiency is preserved. Sufficient conditions which preserve efficiency are found and a numerical example illustrating the results is provided.

**Keywords:** data envelopment analysis, efficiency, simultaneous proportionate change of inputs and outputs, sensitivity analysis, linear programming

## 1. Introduction

Sensitivity analysis in Data Envelopment Analysis (DEA) for the Charnes-Cooper-Rhodes (CCR) ratio model was introduced by Charnes et al. [5] for the case of the change of single output. Sufficient conditions for an efficient Decision Making Unit (DMU) to continue to be efficient after the change of single output were found

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first [5]. The generalizations of that result for the case of the simultaneous change of all outputs, the case of the simultaneous single output and single input changes, the case of the simultaneous change of all inputs and the case of the simultaneous change of all inputs and outputs for the CCR ratio model were given by Charnes and Neralić [6], [7], [8]. Similar results for the additive model were found by Charnes and Neralić [9]. Sufficient conditions for an efficient DMU to preserve efficiency after the proportionate change of inputs (or outputs) were given by Charnes and Neralić [10]. The results in the case of the proportionate change of inputs (or outputs) can be used for ranking among efficient DMUs as it was suggested by Banker and Gifford [2].

The aim of that paper is to study the case of the simultaneous proportionate change (increase) of inputs and proportionate change (decrease) of outputs of an efficient DMU preserving efficiency. Using the results of Charnes and Neralić [8] in sensitivity analysis in DEA for the CCR ratio model sufficient conditions for an efficient DMU to preserve efficiency after the simultaneous proportionate change of inputs and outputs are given.

The paper is organized as follows. The results in sensitivity analysis in DEA which will be used later are contained in Section 2. The main result of the paper is given in the Theorem 2 in Section 3. Section 4 gives an illustrative example. The last Section contains some conclusions.

## 2. Preliminaries

Let us suppose that there are  $n$  Decision Making Units (DMUs) with  $m$  inputs and  $s$  outputs. Let  $x_{ij}$  be the observed amount of  $i$ th type of input of the  $j$ th DMU ( $x_{ij} > 0, i = 1, 2, \dots, m, j = 1, 2, \dots, n$ ) and let  $y_{rj}$  be the observed amount of output of the  $r$ th type for the  $j$ th DMU ( $y_{rj} > 0, r = 1, 2, \dots, s, j = 1, 2, \dots, n$ ). Let  $Y_j, X_j$  be the observed vectors of outputs and inputs of the DMU $_j$ , respectively,  $j = 1, 2, \dots, n$ . Let  $e$  be the column vector of ones and let  $T$  as a superscript denote the transpose. In order to see if the DMU $_{j_0} = \text{DMU}_0$  is efficient according to the CCR ratio model the following linear programming problem should be solved:

$$\min 0\lambda_1 + \dots + 0\lambda_0 + \dots + 0\lambda_n - \varepsilon e^T s^+ - \varepsilon e^T s^- + \theta$$

subject to

$$\begin{aligned} Y_1\lambda_1 + \dots + Y_0\lambda_0 + \dots + Y_n\lambda_n - s^+ &= Y_0 \\ -X_1\lambda_1 - \dots - X_0\lambda_0 - \dots - X_n\lambda_n - s^- + X_0\theta &= 0 \\ \lambda_1, \dots, \lambda_n, s^+, s^- &\geq 0, \end{aligned} \quad (1)$$

with  $Y_0 = Y_{j_0}, X_0 = X_{j_0}, \lambda_0 = \lambda_{j_0}$  and  $\theta$  unconstrained. The symbol  $\varepsilon$  represents the infinitesimal we use to generate the non-Archimedean ordered extension field we

shall use. In this extension field  $\varepsilon$  is less than every positive number in our base field, but greater than zero.  $DMU_0$  is DEA efficient if and only if for the optimal solution  $(\lambda^*, s^{**}, s^{**}, \theta^*)$  of the linear programming problem (1) both of the following are satisfied (for details see [3]):

$$\begin{aligned} \min \quad \theta &= \theta^* = 1 \\ s^{**} &= s^{**} = 0, \text{ in all alternative optima.} \end{aligned} \quad (2)$$

We are interested in variations of all inputs and all outputs of an efficient  $DMU_0$  preserving efficiency. A decrease of any input cannot worsen an already achieved efficiency rating. Downward variations of inputs are not possible in the efficiency rating for an efficient  $DMU_0$ . Hence we can restrict attention to upward variations of inputs of an efficient  $DMU_0$  which can be written as

$$\hat{x}_{i0} = x_{i0} + \beta_i, \quad \beta_i \geq 0, \quad i = 1, 2, \dots, m. \quad (3)$$

Similarly, an increase of any output cannot worsen an already achieved efficiency rating. Upward variations of outputs are not possible in the efficiency rating for an efficient  $DMU_0$ . Hence we can restrict attention to downward variations of outputs which can be written as

$$\hat{y}_{r0} = y_{r0} - \alpha_r > 0, \quad \alpha_r \geq 0, \quad r = 1, 2, \dots, s. \quad (4)$$

For an efficient  $DMU_0$  because of (2) vectors  $[Y_0 - X_0]^T$  and  $[0 \ X_0]^T$  must occur in some optimal basis, which means that there is a basic optimal solution to (1) with  $\lambda_0^* = 1$  and  $\theta^* = 1$ . Changes (3) and (4) are then accompanied by alterations in the inverse  $B^{-1}$  of the optimal basis matrix

$$B = \begin{bmatrix} Y_B & -I_B^+ & 0 & 0 \\ -X_B & 0 & -I_B^- & X_0 \end{bmatrix}, \quad (5)$$

which corresponds to the optimal solution  $(\lambda^*, s^{**}, s^{**}, \theta^*)$  of (1) with  $\lambda_0^* = 1$  and  $\theta^* = 1$ . Let

$$B^{-1} = [b_{ij}^{-1}], \quad i, j = 1, 2, \dots, s+m,$$

be the inverse of the optimal basis  $B$  in (5). Let  $P_j$ ,  $j = 1, 2, \dots, n+s+m+1$  be the columns of the matrix and let  $P_0$  be the right hand side of the linear programming problem (1). We will use the following notations:

$$\begin{aligned} \Gamma_j &= B^{-1}P_j, \quad j = 0, 1, \dots, n+s+m+1, \\ \omega^T &= c_B^T B^{-1}, \\ z_j &= c_B^T B^{-1} P_j \\ &= \omega^T P_j, \quad j = 0, 1, \dots, n+s+m+1. \end{aligned}$$

The simultaneous change of outputs (3) and inputs (4) leads to the following change of the optimal basis matrix  $B$

$$\hat{B} = B + \Delta B \quad (6)$$

with

$$\Delta B = \begin{bmatrix} 0 & \cdots & 0 & \overset{k}{\downarrow} & -\alpha_1 & 0 & \cdots & \overset{s+m}{\downarrow} & 0 \\ 0 & \cdots & 0 & \overset{k}{\downarrow} & -\alpha_2 & 0 & \cdots & \overset{s+m}{\downarrow} & 0 \\ \vdots & & \vdots & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & \overset{k}{\downarrow} & -\alpha_s & 0 & \cdots & \overset{s+m}{\downarrow} & 0 \\ 0 & \cdots & 0 & \overset{k}{\downarrow} & -\beta_1 & 0 & \cdots & \overset{s+m}{\downarrow} & \beta_1 \\ 0 & \cdots & 0 & \overset{k}{\downarrow} & -\beta_2 & 0 & \cdots & \overset{s+m}{\downarrow} & \beta_2 \\ \vdots & & \vdots & & \vdots & \vdots & & & \vdots \\ 0 & \cdots & 0 & \overset{k}{\downarrow} & -\beta_m & 0 & \cdots & \overset{s+m}{\downarrow} & \beta_m \end{bmatrix} \quad (7)$$

and the following change of the right hand side vector

$$\hat{P}_0 = P_0 + [-\alpha_1 \ -\alpha_2 \ \dots \ -\alpha_s \ 0 \ \dots \ 0]^T, \quad (8)$$

where indexes  $k$  and  $s + m$  correspond to the optimal basic variables  $\lambda_0^* = 1$  and  $\theta^* = 1$  respectively. Using matrices

$$U_{(s+m) \times 2} = \begin{bmatrix} \alpha_1 & \alpha_1 \\ \alpha_2 & \alpha_2 \\ \vdots & \vdots \\ \alpha_s & \alpha_s \\ \beta_1 & 0 \\ \beta_2 & 0 \\ \vdots & \vdots \\ \beta_m & 0 \end{bmatrix} \quad (9)$$

and

$$V_{2 \times (s+m)}^T = \begin{bmatrix} 0 & \cdots & 0 & \overset{k}{\downarrow} & -1 & 0 & \cdots & 0 & \overset{s+m}{\downarrow} & 1 \\ 0 & \cdots & 0 & \overset{k}{\downarrow} & 0 & 0 & \cdots & 0 & \overset{s+m}{\downarrow} & -1 \end{bmatrix} \quad (10)$$

we can write the perturbation matrix (7) as  $\Delta B = UV^T$ . Let us use the abbreviation

$$M = I + V^T B^{-1} U,$$

where matrix  $M$  is nonsingular with

$$\begin{aligned} \det M &= 1 - \sum_{t=1}^s b_{k,t}^{-1} \alpha_t + \sum_{t=1}^m (-b_{k,s+t}^{-1} + b_{s+m,s+t}^{-1}) \beta_t + \\ &+ \left( \sum_{t=1}^s b_{s+m,t}^{-1} \alpha_t \right) \left( \sum_{t=1}^m b_{k,s+t}^{-1} \beta_t \right) - \left( \sum_{t=1}^s b_{k,t}^{-1} \alpha_t \right) \left( \sum_{t=1}^m b_{s+m,s+t}^{-1} \beta_t \right), \end{aligned} \quad (11)$$

and

$$D = UM^{-1}V^T. \quad (12)$$

**Theorem 1.** *Conditions*

$$\omega^T D \Gamma_j \geq z_j - c_j, \quad j \text{ an index of nonbasic variables,} \quad (13)$$

are sufficient for  $DMU_0$  to be efficient after the simultaneous changes of inputs (3) and of outputs (4). If  $\det M > 0$ , conditions (13) can be written in the following way

$$\gamma_k \Gamma_{kj} + \gamma_{s+m} \Gamma_{s+m,j} \geq (z_j - c_j) \det M, \quad (14)$$

with

$$\gamma_k = -(1 + \sum_{t=1}^m b_{s+m,s+t}^{-1} \beta_t) (\sum_{t=1}^s \omega_t \alpha_t) + (-1 + \sum_{t=1}^s b_{s+m,t}^{-1} \alpha_t) (\sum_{t=1}^m \omega_{s+t} \beta_t), \quad (15)$$

and

$$\gamma_{s+m} = (\sum_{t=1}^m b_{k,s+t}^{-1} \beta_t) (\sum_{t=1}^s \omega_t \alpha_t) + (1 - \sum_{t=1}^s b_{k,t}^{-1} \alpha_t) (\sum_{t=1}^m \omega_{s+t} \beta_t). \quad (16)$$

For the proof and details see [8].

### 3. Simultaneous proportionate change of inputs and outputs

Let us consider the simultaneous proportionate change (increase) of all inputs

$$\hat{x}_{i0} = \hat{\beta} x_{i0}, \quad \hat{\beta} \geq 1, \quad i = 1, 2, \dots, m, \quad (17)$$

and the proportionate change (decrease) of all outputs

$$\hat{y}_{r0} = \hat{\alpha} y_{r0}, \quad 0 < \hat{\alpha} \leq 1, \quad r = 1, 2, \dots, s, \quad (18)$$

of an efficient  $DMU_0$  preserving efficiency. We are interested in sufficient conditions for  $DMU_0$  to preserve efficiency after the simultaneous changes (17) and (18).

**Theorem 2.** . Let us suppose that  $DMU_0$  is efficient and let

$$\det M = 1 - a_1(1 - \hat{\alpha}) + (-b_1 + b_2)(\hat{\beta} - 1) + (a_2 b_1 - a_1 b_2)(1 - \hat{\alpha})(\hat{\beta} - 1) > 0, \quad (19)$$

with

$$a_1 = \sum_{t=1}^s b_{k,t}^{-1} y_{t0}, \quad a_2 = \sum_{t=1}^s b_{s+m,t}^{-1} y_{t0}, \quad b_1 = \sum_{t=1}^m b_{k,s+t}^{-1} x_{t0}, \quad b_2 = \sum_{t=1}^m b_{s+m,s+t}^{-1} x_{t0}. \quad (20)$$

Let

$$a_3 = \sum_{t=1}^s \omega_t y_{t0}, \quad b_3 = \sum_{t=1}^m \omega_{s+t} x_{t0}, \quad (21)$$

$$d_j = -a_3 \Gamma_{kj} + a_1 \bar{c}_j, \quad e_j = -b_3 (\Gamma_{kj} - \Gamma_{s+m,j}) + (-b_1 + b_2) \bar{c}_j, \quad (22)$$

$$f_j = (a_2 b_3 - a_3 b_2) \Gamma_{kj} + (a_3 b_1 - a_1 b_3) \Gamma_{s+m,j} - (a_2 b_1 - a_1 b_2) \bar{c}_j, \quad (23)$$

$$j = 1, 2, \dots, n + s + m + 1,$$

with  $\bar{c}_j = z_j - c_j$ . Then the conditions

$$d_j(1 - \hat{\alpha}) + e_j(\hat{\beta} - 1) + f_j(1 - \hat{\alpha})(\hat{\beta} - 1) \geq \bar{c}_j, \quad (24)$$

*j* an index of nonbasic variables,

are sufficient for  $DMU_0$  to preserve efficiency after the simultaneous proportionate changes of inputs (17) and of outputs (18).

Proof: First of all let us show that the proportionate changes (17) and (18) are the special cases of the changes (3) and (4) respectively. Using the substitutions

$$\hat{\beta} = 1 + \beta, \quad \beta \geq 0, \quad (25)$$

and

$$\beta_i = \beta x_{i0}, \quad \beta_i \geq 0, \quad i = 1, 2, \dots, m, \quad (26)$$

we can write (17) as

$$\begin{aligned} \hat{x}_{i0} &= x_{i0} + \beta x_{i0} \\ &= x_{i0} + \beta_i, \quad \beta_i \geq 0, \quad i = 1, 2, \dots, m. \end{aligned} \quad (27)$$

It means that the proportionate change of inputs (17) is the special case of the change of inputs (3) with  $\beta_i, i = 1, 2, \dots, m$  in (26) and  $\beta$  in (25). Similarly, if we put

$$\hat{\alpha} = 1 - \alpha, \quad 0 \leq \alpha < 1, \quad (28)$$

and

$$\alpha_r = \alpha y_{r0}, \quad \alpha \geq 0, \quad i = 1, 2, \dots, s, \quad (29)$$

we can write (18) as

$$\begin{aligned} \hat{y}_{r0} &= y_{r0} - \alpha y_{r0} \\ &= y_{r0} - \alpha_r > 0, \quad \alpha_r \geq 0, \quad r = 1, 2, \dots, s. \end{aligned} \quad (30)$$

It means that the proportionate change of outputs (18) is the special case of the change of outputs (4), with  $\alpha_r, r = 1, 2, \dots, s$  in (29) and  $\alpha$  in (28).

Let us suppose that conditions (24) are satisfied. Then using (25), (28), (20)-(23) and (19) it is easy to show that conditions (24) are equivalent to conditions (14) for the case with  $\beta_i, i = 1, 2, \dots, m$  in (26) and  $\alpha_r, r = 1, 2, \dots, s$  in (29). According to Theorem 1 conditions (14) are sufficient for  $DMU_0$  to preserve efficiency after the changes (3) and (4). Because of the equivalency between conditions (14) and (24) for the special case with  $\beta_i, i = 1, 2, \dots, m$  in (26) and  $\alpha_r, r = 1, 2, \dots, s$  in (29), which means the simultaneous proportionate changes of inputs (17) and outputs (18), it follows that conditions (24) are sufficient for  $DMU_0$  to preserve efficiency after the simultaneous proportionate changes of inputs (17) and outputs (18) and completes the proof.

Remark 1. For the case  $\det M < 0$  instead of  $\det M > 0$  in (19), the inequality sign  $\geq$  in conditions (24) should be changed into  $\leq$ .

Remark 2. The system of inequalities (24) together with conditions (17), (18) and (19) for  $\hat{\alpha}$  and  $\hat{\beta}$  gives the area  $\hat{A}_0$  in the plane with the coordinate system  $\hat{\alpha}\hat{\beta}$ . For each point  $(\hat{\alpha}, \hat{\beta})$  in the area  $\hat{A}_0$  efficiency of  $DMU_0$  will be preserved after the simultaneous proportionate changes of inputs (17) and outputs (18).

Remark 3. We can use the area  $\hat{A}_0$  for ranking among efficient DMUs. For example, if for efficient  $DMU_1$  and  $DMU_2$  holds  $\hat{A}_1 > \hat{A}_2$  it can be said that "DMU<sub>1</sub> is relatively more efficient than DMU<sub>2</sub>" because DMU<sub>1</sub> is less sensitive to the simultaneous proportionate change of inputs and outputs preserving efficiency than DMU<sub>2</sub>. The ranking among efficient DMUs can also be based on the proportionate change of inputs (or outputs) as it was suggested by Banker and Gifford [2] and used by Charnes and Nerlić [10].

#### 4. Illustrative example

We will consider the following example taken from [11] with five DMUs, one output, two inputs and data in Table 1.

Table 1  
Data for the example

	DMU <sub>j</sub>	1	2	3	4	5
Output/Input						
$y_{1j}$		2	4	2	3	2
$x_{1j}$		4	12	8	6	2
$x_{2j}$		6	8	2	6	8

We are interested in the efficiency of  $DMU_4$ , with  $X_0 = [6 \ 6]$  and  $Y_0 = [3]$ . In order to see if  $DMU_4$  is efficient, the following linear programming problem should

be solved:

$$\min 0\lambda_1 + 0\lambda_2 + 0\lambda_3 + 0\lambda_4 + 0\lambda_5 - \varepsilon s_1^+ - \varepsilon s_1^- - \varepsilon s_2^- + \theta$$

subject to

$$\begin{aligned} 2\lambda_1 + 4\lambda_2 + 2\lambda_3 + 3\lambda_4 + 2\lambda_5 - s_1^+ &= 3 \\ -4\lambda_1 - 12\lambda_2 - 8\lambda_3 - 6\lambda_4 - 2\lambda_5 - s_1^- + 6\theta &= 0 \\ -6\lambda_1 - 8\lambda_2 - 2\lambda_3 - 6\lambda_4 - 8\lambda_5 - s_2^- + 6\theta &= 0 \\ \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, s_1^+, s_1^-, s_2^- &\geq 0. \end{aligned} \quad (31)$$

The optimal solution of problem (31) is  $\lambda_0^* = \lambda_4^* = 1$ ,  $\theta^* = 1$ ,  $\lambda_1^* = \lambda_2^* = \lambda_3^* = \lambda_5^* = 0$ ,  $s_1^{+*} = s_1^{-*} = s_2^{-*} = 0$  with  $\min \theta = \theta^* = 1$ , which means that  $DMU_4$  is efficient. The optimal basic variables are  $\lambda_3, \lambda_4$  and  $\theta$ . The optimal basis matrix is

$$B = \begin{bmatrix} 2 & 3 & 0 \\ -8 & -6 & 6 \\ -2 & -6 & 6 \end{bmatrix},$$

with the inverse

$$B^{-1} = \begin{bmatrix} 0 & -\frac{1}{6} & \frac{1}{6} \\ \frac{1}{3} & \frac{1}{9} & -\frac{1}{9} \\ \frac{1}{3} & \frac{1}{18} & \frac{1}{9} \end{bmatrix}, \quad (32)$$

and corresponding optimum tableau in Table 2.

Table 2  
Optimum tableau

	$\Gamma_1$	$\Gamma_2$	$\Gamma_3$	$\Gamma_4$	$\Gamma_5$	$\Gamma_6$	$\Gamma_7$	$\Gamma_8$	$\Gamma_9$	$\Gamma_0$
$\lambda_3$	$-\frac{1}{3}$	$\frac{2}{3}$	1	0	-1	0	$\frac{1}{6}$	$-\frac{1}{6}$	0	0
$\lambda_4$	$\frac{8}{9}$	$\frac{8}{9}$	0	1	$\frac{4}{3}$	$-\frac{1}{3}$	$-\frac{1}{9}$	$\frac{1}{9}$	0	1
$\theta$	$-\frac{2}{9}$	$-\frac{2}{9}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$	$-\frac{1}{18}$	$-\frac{1}{9}$	1	1
$z_j - c_j$	$-\frac{2}{9}$	$-\frac{2}{9}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3} + \varepsilon$	$-\frac{1}{18} + \varepsilon$	$-\frac{1}{9} + \varepsilon$	0	1

Let us consider the simultaneous proportionate change (increase) of inputs

$$\hat{x}_{10} = 6\hat{\beta}, \quad \hat{x}_{20} = 6\hat{\beta}, \quad \hat{\beta} \geq 1 \quad (33)$$

and proportionate change (decrease) of output

$$\hat{y}_{10} = 3\hat{\alpha}, \quad 0 < \hat{\alpha} \leq 1, \quad (34)$$

of  $DMU_4$  preserving efficiency. Using (25) - (26) in (33) we get

$$\hat{x}_{10} = 6 + \beta_1, \quad \beta_1 = 6\beta, \quad \beta_1 \geq 0, \quad (35)$$

$$\hat{x}_{20} = 6 + \beta_2, \quad \beta_2 = 6\beta, \quad \beta_2 \geq 0. \quad (36)$$

Similarly using (28)-(29) in (34) we get

$$\hat{y}_{10} = 3 - \alpha_1 > 0, \quad \alpha_1 = 3\alpha, \quad 0 \leq \alpha_1 < 3. \quad (37)$$

Using (32) we have

$$\omega^T = c_B^T B^{-1} = [0 \ 0 \ 1] B^{-1} = [1/3 \ 1/18 \ 1/9]. \quad (38)$$

If we use (32), (38),  $s = 1$ ,  $m = 2$ ,  $k = 2$ ,  $s + m = 3$  and the elements of Table 1 it is easy to get

$$a_1 = 1, \quad a_2 = 1, \quad b_1 = 0, \quad b_2 = 1, \quad a_3 = 1, \quad b_3 = 1. \quad (39)$$

Because of (39) it follows from (19), (22) and (23)

$$\det M = 1 - (1 - \hat{\alpha}) + (\hat{\beta} - 1) - (1 - \hat{\alpha})(\hat{\beta} - 1) > 0, \quad (40)$$

$$d_j = -\Gamma_{2j} + \bar{c}_j, \quad e_j = -(\Gamma_{2j} - \Gamma_{3j}) - \bar{c}_j, \quad (41)$$

and

$$f_j = -\Gamma_{3j} + \bar{c}_j, \quad j = 1, 2, \dots, 9, \quad (42)$$

respectively, with  $\bar{c}_j = z_j - c_j$ . Using (41) and (42) the conditions (24) can be written in the following way

$$(-\Gamma_{2j} + \bar{c}_j)(1 - \hat{\alpha}) + (-\Gamma_{2j} + \Gamma_{3j} - \bar{c}_j)(\hat{\beta} - 1) + (-\Gamma_{3j} + \bar{c}_j)(1 - \hat{\alpha})(\hat{\beta} - 1) \geq \bar{c}_j, \quad (43)$$

$$j = 1, 2, 5, 6, 7, 8.$$

For example, if  $j = 1$  using elements of Table 2 we have from (43)

$$(-8/9 - 2/9)(1 - \hat{\alpha}) + (-8/9 - 2/9 + 2/9)(\hat{\beta} - 1) + (2/9 - 2/9)(1 - \hat{\alpha})(\hat{\beta} - 1) \geq -2/9,$$

or

$$\hat{\beta} \leq 1.25\hat{\alpha}.$$

It is easy to see that the solution set of the system of inequalities (43) together with constraints (33), (34) and (40) is the triangle ABC in the plane with the coordinate system  $\hat{\alpha}\hat{\beta}$  in Figure 1, with A(1, 1.25),

Figure 1 about here

$B(0.8, 1)$  and  $C(1, 1)$ . The redundant constraints are not sketched in Figure 1. For every point  $(\hat{\alpha}, \hat{\beta})$  which belongs to the triangle ABC the efficiency of  $DMU_4$  will be preserved after the simultaneous proportionate change of inputs (33) with the coefficient  $\hat{\beta}$  and proportionate change of output (34) with the coefficient  $\hat{\alpha}$ . The point  $C(1, 1)$  means that there are no changes of inputs and of output, the point  $A(1, 1.25)$  means the maximal proportionate increase of inputs of  $DMU_4$  for 25% preserving its efficiency and the point  $B(0.8, 1)$  means the maximal proportionate decrease of output of  $DMU_4$  for 20% preserving efficiency of  $DMU_4$ . These results of proportionate change of inputs (or output) in that example are the same as in Charnes and Neralić [10], but as can be seen in Figure 1 these changes can not be done simultaneously.

The area  $\hat{A}_4 = 0.025$  of the triangle ABC can be used for ranking  $DMU_4$  among the other efficient DMUs. We can consider  $DMU_3$  and  $DMU_5$  which are efficient too. It is easy to show that in the case of the simultaneous proportionate change of inputs and output of  $DMU_3$  preserving efficiency for the corresponding area holds  $\hat{A}_3 = 0.49982$  (for  $\varepsilon = 0.00001$ ). In the same way it is easy to see that for the efficient  $DMU_5$  holds  $\hat{A}_5 = 0.49982$  (for  $\varepsilon = 0.00001$ ). According to the Remark 3 because of  $\hat{A}_3 = \hat{A}_5 > \hat{A}_4$  it means that "DMU<sub>3</sub> and DMU<sub>5</sub> are relatively more efficient than DMU<sub>4</sub>".

## 5. Conclusions

The simultaneous proportionate change of inputs and proportionate change of outputs of an efficient  $DMU_0$  preserving efficiency in the case of the CCR ratio model in DEA is studied in the paper. Using the results of Charnes and Neralić [8] in sensitivity analysis in DEA for the CCR ratio model sufficient conditions for an efficient  $DMU_0$  to preserve efficiency are established for the case of the simultaneous proportionate increase of inputs and proportionate decrease of outputs. Sufficiency conditions give for each efficient  $DMU_0$  the area which can be used for ranking among efficient DMUs. A numerical example illustrating the results is provided.

The simultaneous proportionate change of inputs with the coefficient  $\hat{\beta}$  and proportionate change of outputs with the coefficient  $\hat{\alpha}$  which is studied can be generalized. For example, the cases of the proportionate change of inputs with different coefficients  $\hat{\beta}_i, i = 1, 2, \dots, m$  or/and the proportionate change of outputs with different coefficients  $\hat{\alpha}_r, r = 1, 2, \dots, s$  can be considered. These cases seem to be interesting also for the BCC model [1] and the additive model [4]. The results for these cases will be presented elsewhere.

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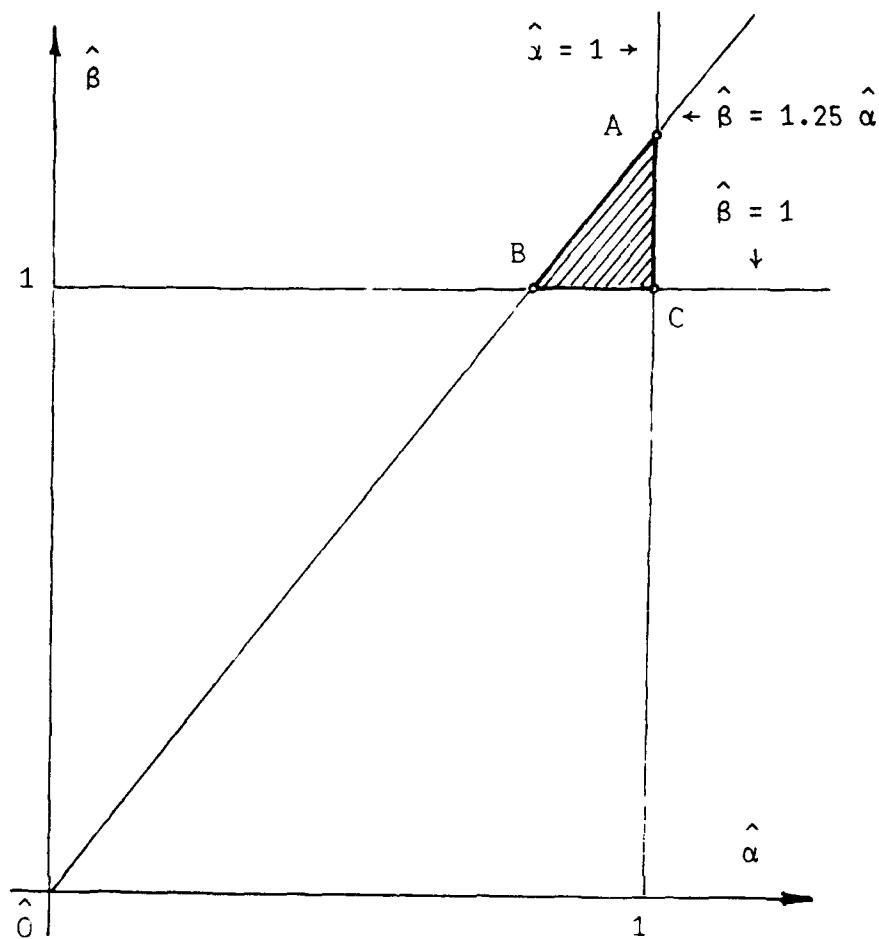


Figure 1. The triangle ABC